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SECOND ORDER LIMIT THEOREMS FOR THE MARKOV BRANCHING PROCESS IN RANDOM ENVIRONMENTS

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In a Markov branching process with random environments, limiting fluctuations of the population size arise from the changing environment, which causes random variation of the 'deterministic' population prediction, and from the stochastic wobble around this 'deterministic' mean, which is apparent in the ordinary Markov branching process. If the random environment is generated by a suitable stationary process, the first variation typically swamps the second kind. In this paper, environmental processes are considered which, in contrast, lead to sampling and environmental fluctuation of comparable magnitude. The method makes little use either of stationarity or of the branching property, and is amenable to some generalization away from the Markov branching process.

Markov branching process	random environments
functional central limit theorem	

1. Introduction

It is proved in [6] that, in a Markov branching process with stationary random environments,

$$X(t)/m(t) \to W,$$

almost surely and in L_2, where $m(t)$ is the expected value of $X(t)$ conditional on the environmental σ-field, provided that the (random) offspring distribution at any time has expected mean greater than one and has finite expected variance. Keiding [7] proves a central limit theorem for $X(t)$ based on this result, by replacing $m(t)$ with $e^{\rho t}$, where ρ is the expected Malthusian parameter, and by using a central limit theorem for stationary processes to circumscribe the fluctuations of $\log m(t) - \rho t$. These fluctuations arise only from the stochastic variation in the environmental process, and are of much larger order than the 'sampling' variation of $X(t)/m(t)$ around W. In this paper, environmental processes are considered which, in contrast, lead to sampling and environmental fluctuation of comparable magnitude.

In Section 2, a functional central limit theorem for $X(t)/\{Wm(t)\}$ is described, in the case when X is a super-critical Markov branching process with deterministic but time-varying parameters. This is then adapted, in Section 3, to random environments. The particular family of environmental processes considered is not very different from those treated in the above references, although neither the branching property nor stationarity are vital to the argument. The paper concludes with some brief remarks on generalization away from the Markov branching process setting.

2. The inhomogeneous Markov branching process

Let $\{X(t), t \geqslant 0\}$ be the minimal continuous time Markov process on the non-negative integers with time-varying Q-matrix given by

$$\begin{aligned} q_{ii}(t) &= -ib(t), \\ q_{ij}(t) &= ib(t)\, p_{j-i}(t), \end{aligned} \qquad i \geqslant 0,\ j \geqslant i-1,\ i \neq j; \tag{2.1}$$

for a formal construction of $X(\cdot)$ see Jacobsen [5], in whose notation

$$\alpha = 0, \quad \beta = \infty,$$

$$G_i]t, \beta] = \exp\left\{-\int_0^t ib(v)\, \mathrm{d}v\right\},$$

$$\pi_{ij}(t) = p_{j-i}(t).$$

It is necessary to assume, for the construction, that $p_j(t) \geqslant 0$ is Borel measurable in $t \geqslant 0$ for each $j \geqslant -1$, with $p_0(t) = 0$ and $\Sigma_j p_j(t) = 1$, and that $b(t) \geqslant 0$ is Lebesgue measurable in $t \geqslant 0$. It will also be assumed that

$$\int_0^t b(u)\, \mathrm{d}u < \infty, \quad 0 \leqslant t < \infty, \tag{2.2}$$

$$\int_0^t b(u)\, \mu(u)\, \mathrm{d}u = M(t) < \infty, \quad 0 \leqslant t < \infty, \tag{2.3}$$

where $\mu(u) = \Sigma_{j=-1}^{\infty} jp_j(u)$, and

$$\int_0^t b(u)\,\sigma^2(u)\,\mathrm{e}^{-M(u)}\,\mathrm{d}u = H(t) < \infty\,, \quad 0 \leqslant t < \infty\,, \tag{2.4}$$

where $\sigma^2(u) = \Sigma_{j=-1}^{\infty} j^2 p_j(u)$. Conditions (2.3) and (2.2) are sufficient to ensure that $X(\cdot)$ is an 'honest' process in $t \geqslant 0$, and that $W(t) \equiv X(t)\,\mathrm{e}^{-M(t)}$ is a non-negative martingale; see, for example, [3] for justification under slightly stronger assumptions. Condition (2.4) implies that

$$\mathrm{E}\{(W(t) - W(s))^2 \mid \mathcal{F}_s\} = W(s)\,(H(t) - H(s))\,,$$

where $\mathcal{F}_s$ denotes the σ-field generated by $\{X(t), 0 \leqslant t \leqslant s\}$. Finally, it is further assumed that

$$\lim_{t\to\infty} H(t) = H < \infty\,; \tag{2.5}$$

then $M(t) \to \infty$ as $t \to \infty$, and $W > 0$ with positive probability, where $W = \lim_{t\to\infty} W(t)$.

Define a sequence of random elements Y_N of $D[0,1]$ by

$$\begin{aligned} Y_N(1) &= \{W(N)\,(H - H(N))\}^{-1/2}\,\{W - W(N)\}\,, \\ Y_N(u) &= \{W(N)\,(H - H(N))\}^{-1/2}\,\{W(t_N(u)) - W(N)\}\,, \\ &\qquad\qquad 0 \leqslant u < 1\,, \end{aligned} \tag{2.6}$$

where

$$t_N(u) = \inf\{t\colon\ t \geqslant N, H(t) - H(N) = u(H - H(N))\}\,. \tag{2.7}$$

If $W(N) = 0$, define

$$Y_N(u) = N^{-1/2}\,B^*(Nu)\,,$$

where $B^*(\cdot)$ is a standard Brownian motion on $[0,\infty)$. Then the following theorem, similar to those of Heyde [4] in the discrete-time case, can be proved.

Theorem 2.1. $Y_N \Rightarrow B$ *in* $D[0,1]$, *where* B *is standard Brownian motion, provided that, for each* $\epsilon > 0$,

$$\lim_{N\to\infty} \left\{ (H - H(N))^{-1} \int_N^\infty b(t)\,\mathrm{e}^{-M(t)} \times \sum_{i} \{j^2 p_j(t)\colon |j| \geqslant \epsilon\,\mathrm{e}^{M(t)} \sqrt{(H - H(N))}\}\,\mathrm{d}t \right\} = 0\,. \tag{2.8}$$

Proof. Condition (2.8) is just the appropriate martingale Lindeberg condition. The theorem can be proved by applying [9, Theorem 2, Condition (B)] to Y_N^*, where $Y_N^*(u) = Y_N(k^{-1}[ku])$ for a suitably large $k = k(N)$. The detail of the proof is straightforward but tedious, and is omitted. □

For the purposes of the next section, it will be convenient to rephrase Theorem 2.1. Let $D^* = D^*[0,1]$ denote the space of left-continuous functions with right limits on $[0,1]$, with the Skorohod topology. Then, because, on $W > 0$, $W/W(N) \to 1$ in probability, and the mapping $\varphi : D[0,1] \to D^*[0,1]$ defined by

$$\varphi(x)(u) = x(1-u) - x(1)$$

is continuous almost surely with respect to Wiener measure, with $B\varphi^{-1} = B$ (on D^*), Theorem 2.1 implies that, on $W > 0$, $Y'_N \Rightarrow B$ in D^*, where

$$Y'_N(u) = \{W/(H - H(N))\}^{1/2}\{W(t_N(1-u))/W - 1\} .$$

Remarking that $|\log(1+x) - x| \leqslant 2x^2$ for $|x| \leqslant \frac{1}{2}$, and that

$$(H - H(N))^{1/2}\{\sup_{0 \leqslant u \leqslant 1}(Y'_N(u))^2\} \to 0 \quad \text{as } N \to \infty,$$

in probability, the following result can be deduced.

Corollary 2.2. *Under the conditions of Theorem 2.1, on $W > 0$, $Z_N \Rightarrow B$ in D^*, where*

$$Z_N(0) = 0 ,$$

$$Z_N(u) = \{W/(H - H(N))\}^{1/2}$$
$$\times \{\log X(t_N(1-u)) - M(t_N(1-u)) - \log W\},$$
$$0 < u \leqslant 1.$$

3. Random environments

In the random environments model, it is assumed that the environmental parameters $b(\cdot)$ and $p(\cdot)$ of Section 2 arise as a realisation of an appropriate stochastic process. Let $\mathcal{G}_t$ denote the σ-field generated by this process up to time t, and denote its underlying probability space by $(\Omega_1, \mathcal{G}, P_1)$, where $\mathcal{G} = \bigvee_t \mathcal{G}_t$. The X process is then a Markov process as in Section 2, with parameters determined by $\mathcal{G}$; let $(\Omega, \mathcal{F}, P)$ denote the overall probability space. As before, if conditions (2.2)–(2.5) hold P_1-a.s., $W = \lim_{t\to\infty} \{X(t)\, e^{-M(t)}\}$ exists P-a.s., and so, on $W > 0$,

$$\log X(t) - M(t) - \log W \to 0 ;$$

the difference now is that $M(t)$ is subject to the random fluctuation of the environmental process. As observed by Keiding [7], if (b, p) is a realisation of a suitably well-behaved stationary process, the fluctuations in $M(t)$ about its mean λt, where $\lambda = \lim_{t\to\infty} \{t^{-1} M(t)\}$, are of order $t^{1/2}$, and completely swamp the declining variability expressed in Corollary 2.2; thus, in Keiding's limit theorem, there is no mention of W. It is, however, possible to obtain more interesting limiting results by modifying the environmental process in such a way that the fluctuations of $M(t)$ are of magnitude comparable to the sampling fluctuation. It would be possible, though not very interesting, to damp the variation of b and p directly, and consequently that of M. Instead, in the example following, the rate of variation is increased, without damping the magnitude, to produce a similar effect on M.

Suppose, then, that (b, p) is a realisation of a stochastic process satisfying the following four conditions:

(i) There exists $\lambda > 0$ such that

$$t^{-\alpha} \{M(t) - \lambda t\} \to 0 \text{ a.s.} \quad \text{as } t \to \infty, \tag{3.1}$$

for some α, $\frac{1}{2} < \alpha < 1$.

(ii) There exists $\nu > 0$ such that

$$(\lambda t)^{-1} \int_0^t b(u)\, \sigma^2(u)\, du \to \nu \text{ a.s.} \quad \text{as } t \to \infty. \tag{3.2}$$

(iii) For each $\epsilon > 0$,

$$t^{-1} \int_0^t b(u) \sum_{|j| \geq \epsilon\sqrt{t}} j^2 p_j(u)\, du \to 0 \text{ a.s.} \quad \text{as } t \to \infty . \tag{3.3}$$

(iv) There exists $\tau > 0$ such that

$$(\lambda\sqrt{\tau})^{-1} M_N \Rightarrow B \text{ on } D^{\alpha}[0, \infty) , \tag{3.4}$$

where $M_N(t) = N^{-1/2}(M(Nt) - N\lambda t)$, independently of $\mathcal{G}$ (cf. [8]).

The space $D^{\alpha}[0, \infty)$ is the set of all right-continuous functions x on $[0, \infty)$ with left limits satisfying $\lim_{t\to\infty} \{t^{-\alpha} x(t)\} = 0$, with the metric m_0 of Whitt [10]; $m_0(x, y)$ is the infimum of $\epsilon > 0$ such that there exists a time deformation $\varphi(\cdot)$ for which

$$\sup_{t\geq 0} \{|x(t) - y(\varphi(t))| / (1 + t^{\alpha})\} \leq \epsilon ,$$

$$\sup_{t\geq 0} \{|\log[(\varphi(t) - \varphi(s))/(t-s)]|\} \leq \epsilon .$$

The mixing stationary processes in [7], for example, would satisfy these conditions. Define new environmental parameters (b^*, p^*) by

$$b^*(t) = b(e^{\lambda t}) , \qquad p^*(t) = p(e^{\lambda t}) , \tag{3.5}$$

and set

$$U_N(0) = 0 ,$$

$$U_N(u) = \sqrt{(W_1\, e^{\lambda N}/(W_1 \tau + \nu))}\, \log(X^*(N - \lambda^{-1} \log u)\, e^{-\lambda N} u / W_1),$$

where $W_1 = \lim_{t\to\infty} \{X^*(t)\, e^{-\lambda t}\}$ and X^* is the new process.

Theorem 3.1. *On $W_1 > 0$, $U_N \Rightarrow B$ in D^*.*

Proof. Let starred quantities relate to the (b^*, p^*) process. Then from (3.1) and (3.2) it follows easily that, P_1-a.s.,

$$\lim_{t\to\infty} \{M^*(t) - \lambda t\} \equiv M_1 = \lambda^{-1} \int_1^{\infty} u^{-1}(b(u)\,\mu(u) - \lambda)\, du < \infty , \tag{3.6}$$

$$\lim_{t\to\infty} \{(H^* - H^*(t)) \exp(\lambda t + M_1)\} = \nu . \tag{3.7}$$

(3.7) implies in turn that

$$\lim_{N\to\infty} \left\{ \sup_{0\leq u<1} \{|t_N^*(1-u) - (N - \lambda^{-1} \log u)|\} \right\} = 0 \quad P_1\text{-a.s.} \tag{3.8}$$

Furthermore, given any $\epsilon > 0$,

$$(H^* - H^*(N))^{-1} \int_N^{\infty} b^*(u)\, e^{-M^*(u)}$$

$$\times \sum \{ j^2 p_j^*(u) \colon |j| \geq \epsilon\, e^{M^*(u)} \sqrt{(H^* - H^*(N))}\}\, du \leq$$

$$\leq \nu^{-1} n \int_n^{\infty} v^{-2}\, b(v) \sum \{ j^2 p_j(v) \colon |j| \geq 2\epsilon_1 v / \sqrt{n}\}\, dv, \tag{3.9}$$

for all N sufficiently large, where $\epsilon_1 = (\epsilon\sqrt{\nu}) \exp(\frac{1}{2} M_1)$ and $n = e^{\lambda N}$. By an argument similar to that in [2, pp. 363–4], (3.3) implies that the right-hand side of (3.9) tends to zero P_1-a.s. as $N \to \infty$. Hence, and from (3.7), given any $\omega_1 \in \Omega_1$ outside a set of P_1-probability zero, Corollary 2.2 implies that, on $W_1 > 0$, $U_N^1 \Rightarrow B$ in D^*, where

$$U_N^1(0) = 0 ,$$

$$U_N^1(u) = \sqrt{(\nu^{-1} W_1 \, e^{\lambda N})} \, \{\log X^*(N - \lambda^{-1} \log u) - M^*(N - \lambda^{-1} \log u) - \log W_1 + M_1\}, \quad 0 < u \leqslant 1.$$

On the other hand,

$$e^{\lambda N/2} \{M^*(N - \lambda^{-1} \log u) - \lambda(N - \lambda^{-1} \log u) - M_1\} =$$

$$= -\lambda^{-1} \, e^{\lambda N/2} \int_{u^{-1} e^{\lambda N}}^{\infty} v^{-1} \{b(v)\, \mu(v) - \lambda\} \, dv$$

$$= \lambda^{-1} \left\{ u M_n(u^{-1}) - \int_{u^{-1}}^{\infty} v^{-2} M_n(v) \, dv \right\} , \tag{3.10}$$

where, as above, $n = e^{\lambda N}$. It now follows, from (3.4), (3.10) and the continuous mapping theorem, that $U_N^2 \Rightarrow B$ in D^*, independently of $\mathcal{G}$, where

$$U_N^2(0) = 0 ,$$

$$U_N^2(u) = \sqrt{(\tau^{-1} \, e^{\lambda N})} \{M^*(N - \lambda^{-1} \log u) - \lambda N + \log u - M_1\} .$$

It only remains to note that

$$U_N(u) = \sqrt{(\nu/(\nu + \tau W_1))} \{U_N^1(u) - \sqrt{(\tau W_1/\nu)} \, U_N^2(u)\} ,$$

and that U_N^1, U_N^2 and W_1 are asymptotically independent on $W_1 > 0$. □

The method used above will go through for environmental schemes other than (3.5), provided that the conditions of Theorem 2.1 can be satisfied. These conditions are not very restrictive; for instance, no stationarity assumption was needed in Section 3 other than the existence of time averages (3.1) and (3.2). Nor does the method depend strongly on the branching property. One can, without too much difficulty, introduce absorbing states other than 0, and positive jump probabilities $p_j(t)$ for $j < -1$. The transition rates q_{ij} can also be taken proportional to i^β, for any β not necessarily equal to 1. In this case, the choice of a suitable martingale is not so direct, but can be accomplished, for example, by using the methods in [1].

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